Section 5.4

The Fundamental Theorem of Calculus I

(1) The Fundamental Theorem of Calculus Part I
(2) The Indefinite Integral
(3) The Net Change Theorem

The Fundamental Theorem of Calculus I (FTC-1)

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is **any** antiderivative of f, that is, $F' = f$.

Example 1: Evaluate $\displaystyle\int_1^3 \left(x^2 - 6\right) dx$.

An antiderivative of $f(x) = x^2 - 6$ is $F(x) = \frac{x^3}{3} - 6x$.

$$\int_1^3 \left(x^2 - 6\right) dx = F(3) - F(1) = \left(\frac{3^3}{3} - 6(3)\right) - \left(\frac{1^3}{3} - 6(1)\right) = -\frac{10}{3}.$$

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Example 2: Evaluate $\displaystyle\int_0^{\pi/2} (2\cos(x) - 4x)dx$.

An antiderivative of $g(x) = 2\cos(x) - 4x$ is $G(x) = 2\sin(x) - 2x^2$, so

$$\int_0^{\pi/2} (2\cos(x) - 4x)dx = \left(2\sin\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{2}\right)^2\right) = 2 - \frac{\pi^2}{2}.$$

FTC-1: Alternative Formulations

We frequently use the notation $F(x)\Big|_a^b$ to stand for $F(b) - F(a)$, so that FTC-1 can be rewritten as

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

FTC-1 says that definite and indefinite integrals are related as follows:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\,\Big|_a^b$$

Example 3: Evaluate $\displaystyle\int_1^4 \frac{5x^2 - \sqrt{x} - 3}{\sqrt{x}}\, dx$.

Solution:

$$\begin{aligned}
\int_1^4 \frac{5x^2 - \sqrt{x} - 3}{\sqrt{x}}\, dx &= \int_1^4 \left(\frac{5x^2}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}} - \frac{3}{\sqrt{x}} \right) dx \\
&= \int_1^4 \left(5x^{3/2} - 1 - 3x^{-1/2} \right) dx \\
&= 2x^{5/2} - x - 6x^{1/2} \Big|_1^4 \\
&= (2 \cdot 32 - 4 - 6 \cdot 2) - (2 - 1 - 6) = \boxed{53.}
\end{aligned}$$

Here is another way of writing FTC-1:

$$\int_a^b F'(x)\,dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a).$$

This version of FTC is often referred to as the **Net Change Theorem**, because it says that the integral of $F'(x)$ — that is, the integral of the rate of change of $F(x)$ — is the net change in F.

Example 4: Water is leaking out of the bottom of a storage tank. The rate of flow at time t (in minutes) is $r(t) = 200 - 8t$ (in L/min). How much water is lost between $t = 5$ and $t = 20$?

Solution: Let $w(t)$ be the amount of water in the tank at time t, so that

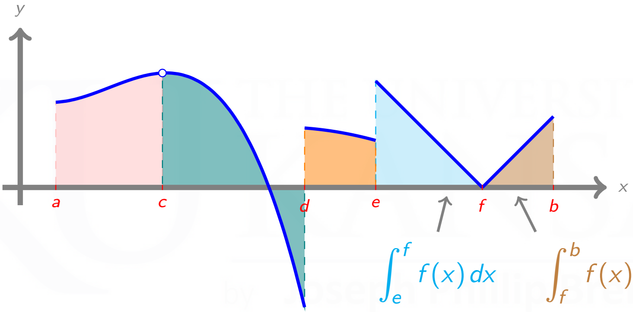
$$w'(t) = -r(t).$$

The net change in the amount of water in the tank is therefore

$$\begin{aligned} w(20) - w(5) &= \int_5^{20} w'(t)\,dt = \int_5^{20} (8t - 200)\,dt \\ &= 4t^2 - 200t\Big|_5^{20} = (4\cdot 20^2 - 200\cdot 20) - (4\cdot 5^2 - 200\cdot 5) \\ &= -1500\,\text{L}. \end{aligned}$$

This calculation produces a negative number because the amount of water in the tank has decreased. The amount of water lost is 1500 L.

The statement of FTC-1 applies only to continuous functions, but in fact it can be used to integrate functions whose only discontinuities are a **finite** number of holes or jumps. For example:



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \int_e^f f(x)\,dx + \int_f^b f(x)\,dx$$

and each the four integrals on the right can be evaluated using FTC.

Warning: FTC-1 cannot be used if $f(x)$ has an infinite discontinuity (vertical asymptote). We will explore this further in MATH 126.

Example 5: The velocity function (in meters per second) for a particle moving along a line is

$$v(t) = 3t - 8$$

(i) Find the displacement of the object from time $t = 0$ to time $t = 4$.
(ii) Find the total distance traveled from time $t = 0$ to time $t = 4$.

(i) The displacement is the net change in position:

$$\int_0^4 v(t)\,dt \;=\; \left(\frac{3t^2}{2} - 8t\right)\Bigg|_0^4 \;=\; \boxed{-8.}$$

(ii) $v(t)$ is negative on $\left(0, \frac{8}{3}\right)$ and positive on $\left(\frac{8}{3}, 4\right)$. The total distance is

$$\begin{aligned}\int_0^4 |v(t)|\,dt \;&=\; -\int_0^{\frac{8}{3}} v(t)\,dt + \int_{\frac{8}{3}}^4 v(t)\,dt \\ &=\; \left(8t - \frac{3t^2}{2}\right)\Bigg|_0^{8/3} + \left(\frac{3t^2}{2} - 8\right)\Bigg|_{8/3}^4 \;=\; \boxed{\frac{40}{3}.}\end{aligned}$$

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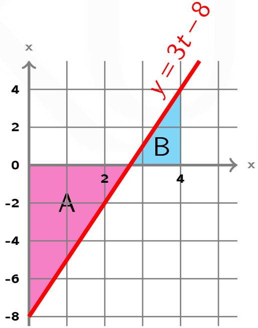

$$\begin{aligned} \text{Displacement} &= \int_0^4 v(t)\,dt \\ &= -A + B \end{aligned}$$

$$\begin{aligned} \text{Total distance traveled} &= \int_0^4 |v(t)|\,dt \\ &= A + B \end{aligned}$$